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ON THE COMPLEX ROOTS OF NUMERICAL EQUATIONS OF THE THIRD AND FOURTH DEGREE.

By A. C. BURNHAM, Berlin, Germany.

The real roots of a numerical equation can, as is well known, be found to any desired degree of accuracy by Horner's method of approximation. The complex roots as well can, for cubic and biquadratic equations, be very easily found by the same method. In fact a single application of Horner's method is in these cases sufficient for determining all the roots to any desired number of decimal places, whether the roots be positive or negative, commensurable or incommensurable, real or complex.

THE CUBIC EQUATION.

Let the cubic equation

$$x^3+a_1x^2+a_2x+a_3=0\ldots\ldots\ldots\ldots\ldots\ldots\ldots(A),$$

have the roots c, $a+bi$, $a-bi$, since one root must be real, where $i=\sqrt{-1}$ and a, b, c are real. The sums of the products of the roots one, two, and three at a time are equal respectively to $-a_1$, a_2, $-a_3$. That is

$$a+bi+a-bi+c=-a_1,$$

$$(a+bi)(a-bi)+(a+bi)c+(a-bi)c=a_2,$$

$$(a+bi)(a-bi)c=-a_3,$$

or

$$2a+c=-a_1\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$a^2+b^2+2ac=a_2\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$$(a^2+b^2)c=-a_3\ldots\ldots\ldots\ldots\ldots\ldots\ldots(3).$$

From these three equations it is not difficult to get the following:

$$8a^3+8a_1a^2+2(a_1{}^2+a_2)a+a_1a_2-a_3=0\ldots\ldots\ldots\ldots\text{I}.$$

$$c=-(a_1+2a)\text{ or }a=-\tfrac{1}{2}(c+a_1)\ldots\ldots\ldots\ldots\text{II}.$$

$$\pm\sqrt{\frac{a_3-a_1a^2-2a^3}{2a+a_1}}=b=\pm\sqrt{\frac{-a_3}{c}-a^2}=\pm\sqrt{\frac{-a_3-\frac{1}{4}c(c+a_1)^2}{c}}\ldots\ldots\text{III}.$$

Equations I, II, III give all the roots to any desired degree of accuracy. One may find c from the given equation (A) by Horner's method, or a from equation I by the same method, according to which is the easier. The a or c and b are given by II and III by a mere substitution. It is, of course, immaterial whether the positive or negative value of b be taken, since, in any case, both are used. b will be imaginary only when the original equation (A) has all three roots real. It is also of no consequence which of the three values for a given by equation I be taken, but I will in no case have a greater number of real roots than the given equation (A).

EXAMPLE. Find the roots of $x^3-2x-5=0$.

The one real root c, easily found by Horner's method is, $c=2.0945+$.

We have moreover, $a_1=0$, $a_2=-2$, $a_3=-5$. Therefore, $a=-\frac{1}{2}(c+a_1)$ $=-1.0472+$, and $b=\sqrt{\frac{5}{2.0945}-(-1.0472)^2}=1.123$.

The roots therefore are $-1.0472\pm1.123\sqrt{-1}$ and 2.0945.

In this example, equation I. takes the form

$$8a^3-4a+5=0,$$

which has the one real root $a=1.0472+$. This is the same result as above.

THE BIQUADRATIC EQUATION.

Let the roots of the biquadratic equation

$$x^4+a_1x^3+a_2x^2+a_3x+a_4=0 \quad\ldots\ldots\ldots\ldots\ldots\ldots (B).$$

be $a\pm bi$, $c\pm di$. We then have as before

$$a+bi+a-bi+c+di+c-di=-a_1,$$

$$(a+bi)(a-bi)+(a+bi)(c+di)+(a+bi)(c-di)+(a-bi)(c+di)$$
$$+(a-bi)(c-di)+(c+di)(c-di)=a_2,$$

$$(a+bi)(a-bi)(c+di)+(a+bi)(a-bi)(c-di)+(a+bi)(c+di)(c-di)$$
$$+(a-bi)(c+di)(c-di)=-a_3,$$

$$(a+bi)(a-bi)(c+di)(c-di)=a_4,$$

or

$$2(a+c)=-a_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

$$(a^2+b^2)+(c^2+d^2)+4ac=a_2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2).$$

$$2c(a^2+b^2)+2a(c^2+d^2)=-a_3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

$$(a^2+b^2)(c^2+d^2=a_4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

From these four equations we find

$$a=-\tfrac{1}{2}(2c+a_1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

$$c=-\tfrac{1}{2}(2a+a_1)\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6).$$

$$c^2+d^2=u=\frac{2ca_2+a_3+4c^2(2c+a_1)}{4c+a_1}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{I}.$$

$$a^2+b^2=t=-\frac{2au+a_3}{2c}=\frac{a_4}{u}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{II}.$$

If we now eliminate u and a by means of 5, I and II, we have the following equation of the sixth degree for c:

$$\begin{aligned}64c^6+96a_1c^5+16(3a_1{}^2+2a_2)c^4+8(4a_1a_2+a_1{}^3)c^3&\\ +4(a_2{}^2+2a_1{}^2a_2+a_1a_3-4a_4)c^2+2(a_1a_2{}^2+a_1{}^2a_3-4a_1a_4)c&\\ +a_1a_2a_3-a_1{}^2a_4-a_3{}^2=0\ldots\ldots\ldots\ldots\ldots\ldots\text{III}.&\end{aligned}$$

From I and II we have moreover,

$$d=\pm\sqrt{u-c^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7).$$

$$b=\pm\sqrt{t-a^2}\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8).$$

Therefore, after getting a single value of c from III by Horner's method, a, u, t, d, and b follow respectively from (5), I, II, (7), and (8) by mere substitutions, and thus a single application of Horner's method suffices to find all of the roots, no matter what their character.

If the equation (B) has no real roots, then III will have only two real roots. They are separately the values for a and c, and either can be taken for c. That is, the equation of the sixth degree giving a is the same as III giving c.

EXAMPLE. Find the roots of the equation

$$x^4-6x^3+18x^2-30x+25=0.$$

In this equation $a_1=-6$, $a_2=18$, $a_3=-30$, $a_4=25$.
We have therefore as equation III,

$$4c^6-36c^5+144c^4-324c^3+425c^2-303c+90=0.$$

It is immediately seen that one is a root of this equation, therefore $c=1$, from which there follows,

from (5), $a=-\frac{1}{2}(2-6)=2$,

from I, $u=[2.18-30+4(2-6)]/[-2]=5$,

from II, $t=a_4/u=\frac{25}{5}=5$,

from (7), $d=\pm\sqrt{5-1}=\pm2$,

from (8), $b=\pm\sqrt{5-4}=\pm1$.

The roots are therefore $2\pm i$, $1\pm2i$.

The value 2 for a satisfies the equation III as it should, and 1 and 2 are the only real roots which III possesses.

If the given equation (B) has two real roots and both are known to any desired degree of accuracy, the two other roots are very easily found. Put

$$a+bi=h$$
$$a-bi=k$$

where h and k are known. Then

$$a=\tfrac{1}{2}(h+k),$$
$$b=-\tfrac{1}{2}(h-k)i,$$
$$c=-\tfrac{1}{2}(h+k+a_1) \text{ from (5)},$$

and u is found from I and d from (7) as before. Thus the roots are all determined.

If all of the roots of (B) are real, they will be equally well given by the first method above. In this case b and d will be imaginary.

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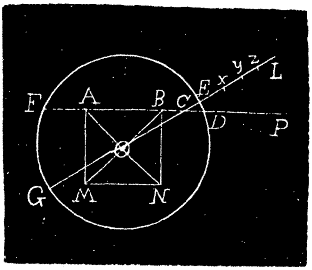
A DEVICE FOR EXTRACTING THE SQUARE ROOT OF CERTAIN SURD QUANTITIES.

By ROBERT J. ALEY, A. M., Ph. D., Professor of Mathematics, University of Indiana, Bloomington, Indiana.

ABMN is a square. *OL* is an arm revolving freely about *O*. This arm beyond *C* is divided into equal parts at *E*, *x*, *y*, *z*, etc.



To determine the character of the divisions made on *FP* by the points of division on *OL* as *OL* revolves. Call the side of the square *AB*, $2a$; *BC*, b ; *CE*, c ; and *CD*, x.

Then $OC=\sqrt{a^2+(a+b)^2}$.

$$GC=2\sqrt{a^2+(a+b)^2}+c.$$